The Optical Tunnelling Problem For Fibres With W-Shaped Refractive Index Profiles

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I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Master in Applied Mathematical Sciences is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Abstract

In modelling any physical situation, a balance must be struck between making enough assumptions to give a mathematically tractable problem, yet sufficiently few assumptions for the model to remain physically realistic. In this thesis we consider three models which have been proposed for radiation losses in bent fibre optic waveguides and put forward a model of our own which will accommodate a fibre with a W-shaped refractive index profile of a type currently encountered in industrial production.

In Chapter 1 we review the original model of Kath and Kriegsmann. We describe in Chapter 2 an idealised ordinary differential equation model due to Paris and Wood and its adaptation by Burzlaff and Wood to step-function profiles. Neither of these idealised models will handle the the realistic W-shaped profile. Chapter 3 contains new work whereby, following the approach of Burzlaff and Wood, we construct a model which incorporates in the boundary condition various geometrical parameters describing the W-shaped profile. The exponentially small imaginary part of the eigenvalue of the resulting boundary value problem corresponds physically to the rate of radiation loss from the fibre.

To solve this problem we use a new general method of Hu and Cheng, which relies on concepts introduced by Gingold. These are outlined in Chapter 4. Chapter 5 starts with a rederivation of the formula of Hu and Cheng for the imaginary part of the eigenvalue for general potentials. For our model of Chapter 3 the method of Hu and Cheng can be simplified and we obtain an asymptotic estimate of the eigenvalue based on Hankel function solutions of the differential equation. This is the main result of the thesis. We conclude by showing that the general formula of Hu and Cheng yields the correct approximation for the rate of radiation loss in the power index models considered by Brazel, Lawless, Liu and Wood.

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Chapter 1

The Physical Problem.

In this, our opening chapter, we will briefly introduce the optical tunnelling problem developed by Kath and Kriegsmann [9] in their 1987 paper "Optical Tunnelling Radiation losses in bent fibre optic waveguides" We will outline the various assumptions the authors make, and show how making use of these assumptions, the authors were able to develop a scalar wave equation for their model and reduce it to a simple form Finally, we will give a brief physical explanation of their model and its results

1.1 Kath and Kriegsmann's Optical Tunnelling Problem.

Kath & Kriegsmann [9] recently considered radiation losses in bent fibre-optic waveguides, where arbitrary deformations including torsion, in three dimensions were permitted Such waveguides are unable to trap light perfectly and as a result, energy slowly tunnels out of the core region and radiates away into the cladding This rate of energy loss is represented, in their mathematical model, by an exponentially small imaginary part of a complex eigenvalue λ of a differential equation boundary value problem

Now, they assume at the start of their paper that the radius of curvature of the bent fibre is very large compared to the wavelength of light used Also, they take the fibre to be weakly guiding, so that the refractive index in the cladding deviates only slightly from that in the core This, in particular, is a justifiable approximation for fibres which support a small number of guided modes, and especially for monomode fibres As a result of these assumptions, they felt justified in making use of the parabolic or paraxial approximation, (p103, [12]) in their model

So, the authors started off by constructing a suitable coordinate system, one which followed the centre-line of the fibre, whilst taking curvature and torsion into account and scaled it in terms of the radius a of the inner core They made use of the fact that for weakly guiding fibres a scalar theory is a reasonable approximation, (p339, [12]), and obtained, in a straightforward manner, a scalar approximation, directly from Maxwell's equations

They achieved this, by starting with the curl version of the time–harmonic wave equation for the electric field, given by

$$\nabla \times (\nabla \times \mathbf{E}) - \mathrm{n}^2\mathrm{k}^2\mathbf{E} = \mathbf{0} \qquad (1\ 1\text{-}1)$$

where

$$\begin{aligned}
\mathbf{E} &= \text{Electric field,} \\
\mathrm{n}_0 &= \text{Core refractive index,} \\
\mathrm{n}_c &= \text{Cladding refractive index,} \\
\mathrm{n} &= \text{Normalised refractive index,} \\
\mathrm{a} &= \text{Core radius,} \\
\mathrm{k}_0 &= \text{Physical wave number,} \\
\mathrm{k} &= \mathrm{k}_o\,\mathrm{a}\,\mathrm{n}_c
\end{aligned}$$

They assumed (1 1-1) to be in a dimensionless form, and took k to be a dimensionless wave number, composed of the physical wave number k_o and the cladding index, n_c n, the normalised index of refraction, $(\mathrm{n} = \mathrm{n}_0/\mathrm{n}_c)$, was taken to be 1 outside the core region Next, they examined typical values, given in [17], for a monomode fibre They found $\mathrm{k} \approx (15 - 40)$, (that is $\mathrm{k}_o \approx 6 \times 10^4\,\mathrm{cm}^{-1}$, $\mathrm{a} \approx 2 - 5\,\mu\mathrm{m}$, $\mathrm{n}_c \approx 1\ 3$)

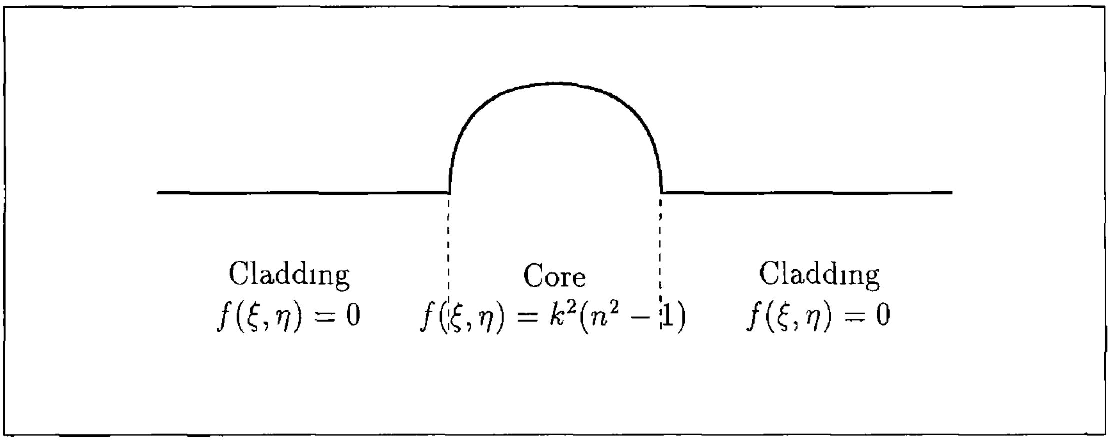

Figure 1 1 Schematic of the behaviour of $f(\xi,\eta)$

So, obviously in what follows, k is treated as a large parameter Also in [17], the authors found typical values in the core region to be $n^2 \approx (1\ 005 - 1\ 02)$ Now, making use of the weakly guiding approximation, which allowed them to assume that the difference between the refractive index in the core and the refractive index in the cladding was negligible, the authors suggested that the correct scaling should be given by (cf Fig 1 1)

$$n^2 = 1 + \frac{f(\xi,\eta)}{k^2}, \tag{1 1-2}$$

where

$f(\xi,\eta)$ = Refractive index in the core relative to the cladding,

(ξ,η) = Coordinate axis orthogonal to the fibre in a torsion free comoving coordinate system

They felt justified in making this scaling as $f(\xi,\eta)$ is usually confined to the range 1 to 5 and is taken to be non-zero only in the core region Thus, the calculated values of k in the range 15 to 40 then gives roughly the right values for n in the core The authors next made use of the parabolic or paraxial approximation [12] and let

$$\mathbf{E} = \mathbf{A}(\sigma,\xi,\eta)\, e^{iks}, \qquad \sigma = \frac{s}{k}, \tag{1 1-3}$$

where the variable σ measures a scaled length along the fibre Substituting (1 1-3) into (1 1-1), they obtained an expression for the amplitude $\mathbf{A}$ of the transverse component of the electric field, described by the equation

$$2i\mathbf{A}_\sigma + \mathbf{A}_{\xi\xi} + \mathbf{A}_{\eta\eta} + f(\xi,\eta)\mathbf{A} + 2\,\mathrm{k}^2\,\delta\,\kappa\,\alpha\,\mathbf{A} + O(\mathrm{k}^2\,\delta^2, \delta\,, 1/\mathrm{k}^2) = \mathbf{0} \qquad (1\ 1\text{-}4)$$

Here $\delta = a/l$, where l is a typical length scale for the bent centreline

Then, assuming that the curvature produces an effect comparable with the scaled index of refraction difference $f(\xi,\eta)$, they took $\mathrm{k}^2\delta$ to be equal to unity This combined with $\mathrm{k} \approx (15 - 40)$ gave them a dimensionless radius of curvature of the order of a few millimetres which proved to be too small Therefore, to give them a radius of curvature of a few centimetres to a few tens of centimetres, the authors assumed that $\delta = 1/\mathrm{k}^3$ Now, with this choice for δ and neglecting all of the small terms, $O(1/\mathrm{k}^2)$ and smaller, Kath and Kriegsmann [9] obtained

$$2i\mathbf{A}_\sigma + \mathbf{A}_{\xi\xi} + \mathbf{A}_{\eta\eta} + f(\xi,\eta)\mathbf{A} + \,(2\,\kappa\,\alpha/\mathrm{k})\,\mathbf{A} = \mathbf{0}, \qquad (1\ 1\text{-}5)$$

where $\alpha = \xi\cos\theta - \eta\sin\theta$ and κ is a scaled curvature which is $O(1)$ Here θ is the rotation of the fibre which removes the torsion As they were looking for solutions to (1 1-5), of the form[1]

$$\mathbf{A}(\sigma,\xi,\eta) = y(\xi,\eta)e^{i\Lambda\sigma}, \qquad (1\ 1\text{-}6)$$

they substituted (1 1-6) into (1 1-5) to get

$$\nabla^2 y + f(\xi,\eta)\,y - \,2\,\Lambda\,y + \,(2\,\kappa\,\alpha/\mathrm{k})\,y = \mathbf{0} \qquad (1\ 1\text{-}7)$$

Finally, to simplify (1 1-7) they let

$$\epsilon = 2\kappa/\mathrm{k} \qquad \text{and} \qquad \lambda = -2\Lambda, \qquad (1\ 1\text{-}8)$$

[1] Λ is basically the difference between the propagation constant of the mode and k_0, also the decay rate $\mathrm{Im}\Lambda$ (which must be positive) implies $\mathrm{Im}\lambda$ must be negative

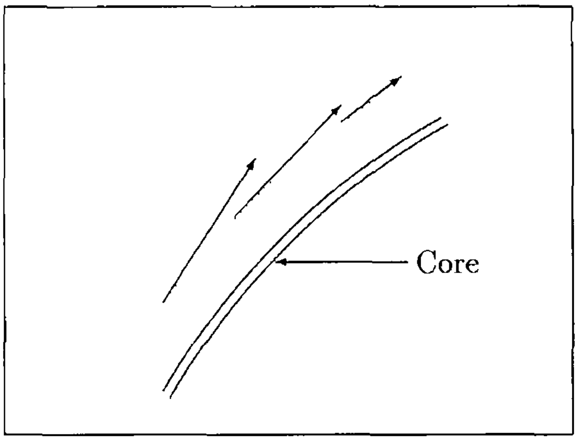

Figure 1 2 This schematic shows energy shedding out of the core region

to get

$$\nabla^2 y + f(\xi,\eta)y + \lambda y + \epsilon\alpha y = \mathbf{0}, \tag{1 1-9}$$

their scalar wave equation, incorporating the profile of their slightly, bent curved optical fibre

1.2 Physical Explanation.

We will now give a brief physical explanation of Kath & Kriegsmann's model, and its results From (1 1-5), we can see that, after the various approximations have been made, the only effect of the curvature on their model is to introduce the term $\epsilon\alpha y$ in (1 1-9), which leads to a 'tilting' of the refractive index $f(\xi,\eta)$ and enables the mode to tunnel out to one side In the cladding, where α is large, $f(\xi,\eta) = 0$ This perturbation curvature can be explained, by viewing the situation in normal Cartesian co-ordinates, (cf Fig 1 2)

In this coordinate system, we can see that energy for large positive values of α (out in the evanescent tail of the mode), must travel further than energy propagating in the core region On transforming to the local coordinate system following the fibre, the influence of this extra distance, is changed to an effective slowing of the wave, via an increased index of refraction

This loss of energy in the mode, can be explained as follows As one moves away from the core, eventually a point is reached, where the energy propagating in the evanescent tail, cannot keep up with the main part of the wave propagating in the core and thereby changes from an evanescent to a propagating wave This energy is then shed as it radiates away into the cladding Of course, because this happens in the evanescent part of the mode the energy loss is not dramatic, but over a lengthy run can be significant If we borrow some terminology from quantum mechanics [10], [1], we can say, that energy tunnels out of the core region, across a potential barrier where the wave is evanescent, until it reaches a region where it can propagate again and radiate away

Chapter 2

Review of Idealised Models.

We will now move on from Kath & Kriegsmann's model and show how Paris & Wood [15], put forward a more idealistic, one-dimensional model, describing Kath & Kriegsmann's fibre. Although not entirely realistic physically, this model had the important advantage of being solvable explicitly in terms of Airy functions whose asymptotic properties are known. This simpler model provided insight into the nature of the problem and formed a basis for future improvements. Burzlaff & Wood [4] modified Paris & Wood's model to produce a less idealised model, which could still be handled using special functions. We will outline how the above were derived, prove their non-self-adjointness, show their profiles and discuss their usefulness. Finally, we will state, for reference purposes, the results obtained by various authors for the radiation loss after solving Kath & Kriegsmann's model (1.1-9) with $\varepsilon\alpha y$ replaced by $\varepsilon\alpha y^n$, for different values of n, n $\in$ Z^+.

2.1 Paris and Wood's Model Problem.

Kath and Kriegsmann proceeded to solve (1.1-9) of the previous chapter, by using singular perturbation theory. They did so by obtaining a regular perturbation expansion in the inner core region and a WKB expansion in the outer cladding. They then proceeded to match these two expansions together and make use of an integral conservation law, to give them an expression for the radiation loss from the fibre. However, although this method is informative in its own right, it does not offer us a completely rigorous analysis of the relatively complicated equation

$$\nabla^2 y + f(\xi,\eta)y + \lambda y + \varepsilon\alpha y = 0 \tag{2 1-1}$$

To do so fully, would prove to be very difficult Therefore, it is desirable instead to study first a simpler one-dimensional model This can be solved explicitly and understood in detail Such a model was proposed in 1991, by Paris and Wood [15] They observed that for small ε, in Kath and Kriegsmann's model, we are located in the cladding region, where the perturbation $f(\xi,\eta)$ in the refractive index is zero Therefore, our interest lies solely in the behaviour of solutions in the neighbourhood of a turning (or transition) point, which is situated well into the cladding region

Thus, Paris & Wood [15] felt justified in considering the model problem,

$$i\phi_t = -\phi_{xx} + \varepsilon\,\mathrm{g}(x)\,\phi, \tag{2 1-2}$$

with the general, homogeneous, boundary condition

$$\phi_x(0,t) + h\,\phi_x(0,t) = 0, \tag{2 1-3}$$

here, $\mathrm{g}(x) = x$, and the positive constant h is twice the integral of the refractive index $f(x)$ over the core region, as will be explained later

One can clearly see that this has the same structure as Kath and Kriegsmann's model So, making the same separation of variables

$$\phi(x,t) = e^{i\lambda t}y(x), \qquad \mathrm{Im}\lambda < 0, \tag{2 1-4}$$

the authors got

$$y''(x) + (\lambda + \varepsilon x)y(x) = 0, \tag{2 1-5}$$

with boundary condition

$$y'(0) + h\, y(0) = 0, \tag{2 1-6}$$

at the origin

Now, to obtain the boundary condition at infinity, the authors had to return to the physical discussion in Section 1 2, which indicates that the solution, must be an outgoing wave beyond the turning point at $x = -\lambda/\varepsilon$ They expressed this condition by constructing any solution $y(x)$ to have controlling behaviour of the form $e^{ip(x)}$, where $p(x)$ is a positive function of x, as $x \to +\infty$ The unique function $p(x)$, is found via the Liouville-Green substitution $y(x) = e^{ip(x)}$ to be $p(x) = (2/3)\,\varepsilon^{1/2}\,x^{3/2}$

Thus their model problem, is given by

$$y''(x) + (\lambda + \varepsilon x)y(x) = 0 \tag{2 1-7}$$

$$y'(0) + hy(0) = 0 \tag{2 1-8}$$

where $y(x)$ has controlling behaviour $e^{ip(x)}$, $p(x) > 0$ as $x \to +\infty$, h is a positive constant and $\varepsilon > 0$

2.2 Proof of Non-Self-Adjointness.

In this section we will show that (2 1-7) is non-self-adjoint, and thus may possess non-real eigenvalues (2 1-7) appears at first glance to be self-adjoint, in which case the spectrum would be real, but a careful analysis shows that the conditions for self-adjointness are broken by the boundary condition at infinity To see this, let $Lu \equiv -u'' - \varepsilon x u$ and denote by $\langle\ ,\ \rangle$ the usual inner product in the Hilbert space $L^2(0,\infty)$ Therefore, for any functions $u, v \in L^2(0,\infty)$ satisfying the boundary conditions, an integration by parts shows that

$$\langle Lu, v\rangle = -[u'(x)\overline{v(x)}]_0^\infty + \int_0^\infty u'(x)\overline{v'(x)}dx - \varepsilon \int_0^\infty xu(x)\overline{v(x)}dx, \tag{2 2-1}$$

$$\langle u\,,\mathrm{L}v\rangle = -[u(x)\overline{v'(x)}]_0^{\infty} + \int_0^{\infty} u'(x)\overline{v'(x)}dx - \varepsilon\int_0^{\infty} xu(x)\overline{v(x)}dx \qquad (2\,2\text{-}2)$$

The self–adjointness condition $\langle \mathrm{L}u, v\rangle = \langle u\,, \mathrm{L}v\rangle$ holds if and only if the integrated terms are equal Since they are clearly equal at the origin, this condition is equivalent to

$$\lim_{x\to\infty} u'(x)\overline{v(x)} = \lim_{x\to\infty} u(x)\overline{v'(x)} \qquad (2\,2\text{-}3)$$

When we insert the proposed outgoing wave behaviour $\exp[ipx]$, we find that we get $u'(x)\overline{v(x)} \sim ip'(x)$, but $u(x)\overline{v'(x)} \sim -ip'(x)$, as $x \to +\infty$ The problem is thus non-self-adjoint and nonreal eigenvalues may occur

Analysing (2 1-7), we can see that it has an eigenvalue at $\lambda = -h^2$, when $\varepsilon = 0$, but the perturbed problem is non-self-adjoint, because of the form of the boundary condition at infinity It is thus possible, for the eigenvalue of the perturbed problem to be non-real, and one can see from [19], that it has an imaginary part which is $O\left(e^{-1/\varepsilon}\right)$ as $\varepsilon \to 0+$

Also, using regular perturbative methods, we can obtain the asymptotic series[1]

$$\lambda = -h^2 - \frac{\varepsilon}{2h} - \frac{\varepsilon^2}{8h^4} - \frac{5\varepsilon^3}{32h^7} - \frac{11\varepsilon^4}{32h^{10}} + O(\varepsilon^5) \qquad (2\,2\text{-}4)$$

Although this can be continued to as high an order as desired, it will never yield any information on $\mathrm{Im}\lambda$ This is hardly surprising, since $\mathrm{Im}\lambda$ turns out to be $o\left(\varepsilon^n\right)$ as $\varepsilon \to 0+$ for any $n \in N$

2.3 Burzlaff and Wood's Model Problem.

In 1991, Burzlaff & Wood [4], considered Paris & Wood's model problem and proposed a more accurate model They considered (2 1-7) to be the limit as $p \to 0$ of

[1] see p31,[14]

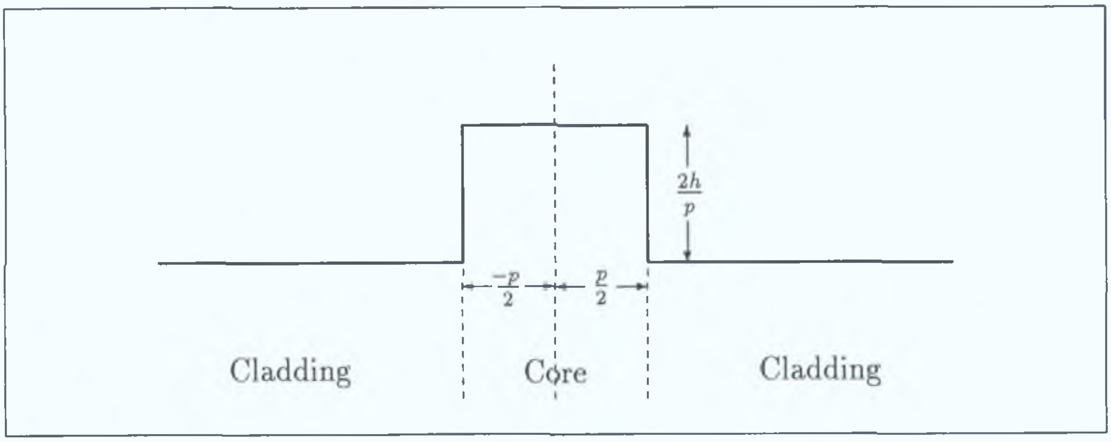

Figure 2.1: Step index profile of Burzlaff and Wood's model.

the following problem:

$$-y''(x) + v(x)y(x) = \lambda y(x), \tag{2.3-1}$$

where

$$v(x) = \begin{cases} -\varepsilon \mid x \mid, & \text{when } \mid x \mid \geq p/2 \\ -2h/p - \varepsilon \mid x \mid, & \text{when } \mid x \mid \leq p/2, \end{cases} \tag{2.3-2}$$

$x \in \Re$ and $y \in \mathrm{C}^1(\Re)$.

They based their model on a step-index profile of a monomode fibre (cf. Fig. 2.1).

Now, reflection symmetry in (2.3-1) implies that the lowest eigenfunction is even and its derivative is odd, so that the authors could restrict their attention to $x \in (0, \infty)$. In the limit as p tends to zero, they obtained a delta-function potential at $x = 0^+$ and the jump condition (2.1-8) for the derivative of y.

Although Paris & Wood's model shares some features with the Kath & Kriegsmann's model, it clearly lacks others. First of all, the optical tunnelling problem is not symmetric (radiation goes out to one side only), thus the problem should be confined to the half–plane. Secondly, the approximation for a weakly guiding fibre is obviously very poor for a delta function potential and thirdly, the optical tunnelling problem is a two-dimensional problem.

In reply to these criticisms, Burzlaff & Wood presented a more realistic one-dimensional

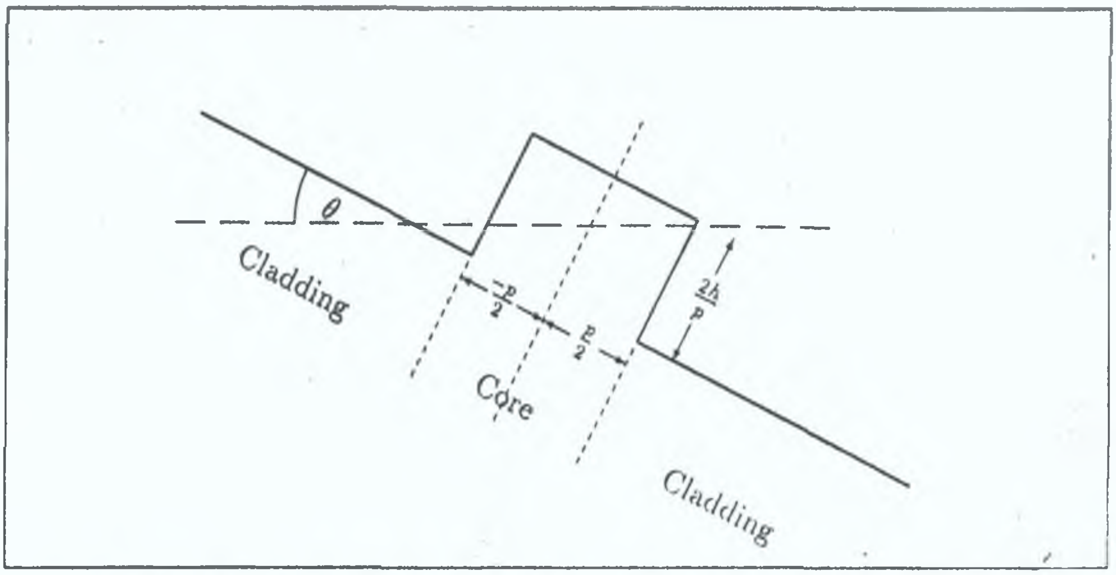

Figure 2.2: Burzlaff and Wood's model profile for a bent fibre.

model, to which the first two criticisms do not apply and in response to the third deficiency, the study of a one-dimensional model, they argued that it could be justified by the physical fact that radiation mainly goes out in a narrow cone, along the "plane of curvature". If that plane does not change much (low torsion) then we have essentially a one-dimensional problem.

So, the following model (cf. Fig. 2.2)[2] replaces (2.3-1)-(2.3-2):-

$$-y''(x) + v(x)y(x) = \lambda y(x), \tag{2.3-3}$$

where

$$v(x) = \begin{cases} -\varepsilon x, & \text{when } \mid x \mid\geq p/2 \\ -2h/p - \varepsilon x, & \text{when } \mid x \mid\leq p/2, \end{cases} \tag{2.3-4}$$

$x \in \Re$ and $h \geq 0$.

Here p is assumed to be much larger than ε, but otherwise arbitrary. In particular, p may be small, so that the weakly guiding approximation is justified. To complete the model, appropriate boundary conditions are chosen at plus and minus infinity. The Paris & Wood model can be obtained by integrating (2.3-1), across the core and

[2] θ is the rotation of the fibre which removes the torsion.

getting its limit as $p \to 0$ However, as the model is now confined to the half-axis, the authors had to impose the conditions $y(x) = y(-x)$ and $y'(x) = -y'(x)$, into their boundary conditions to ensure this is taken into account So we get the limit as $p \to 0$ of

$$\begin{aligned} -\int_{-p/2}^{+p/2} y''(x)\,dx \quad &- \quad 2\,h\int_{-p/2}^{+p/2} (y(x)\,/\,p)\,dx \\ &+ \quad \varepsilon\int_{-p/2}^{+p/2} x\,y(x)\,dx \;=\; -\,\lambda\int_{-p/2}^{+p/2} y(x)\,dx \end{aligned} \tag{2 3-5}$$

Now,

$$\lim_{p\to 0}\int_{-p/2}^{+p/2} (y(x)\,/\,p)\;dx \;=\; \int_{0^-}^{0^+} y(x)\,\delta(x)\,dx, \tag{2 3-6}$$

$\Rightarrow$

$$\begin{aligned} -\int_{0^-}^{0^+} y''(x)\,dx \quad &- \quad 2\,h\int_{0^-}^{0^+} y(x)\,\delta(x)\,dx \\ &+ \quad \varepsilon\int_{0^-}^{0^+} x\,y(x)\,dx \;=\; -\,\lambda\int_{0^-}^{0^+} y(x)\,dx, \end{aligned} \tag{2 3-7}$$

$\Rightarrow$

$$-\,2y'(0^+) - 2hy(0^+) = 0, \tag{2 3-8}$$

$\Rightarrow$

$$y'(0^+) + hy(0^+) = 0, \tag{2 3-9}$$

is their boundary condition at the origin The boundary condition at infinity is the same as in Paris & Wood's model

So as $p \to 0$, the model (2 3-1), (2 3-2) leads to

$$-y''(x) + v(x)y(x) = \lambda y(x), \qquad x \in [0,\infty), \tag{2 3-10}$$

with boundary condition

$$\mathrm{y}'(0^+) + h\mathrm{y}(0^+) = 0 \tag{2 3-11}$$

at the origin, where $x \in [0,\infty)$ and $y \in \mathrm{C}^0(\Re)$ $y(x)$, as discussed, also has the behaviour $e^{ip(x)}$, where $p(x)$ is a positive function of x, as $x \to +\infty$

2.4 Results Obtained From Paris and Wood's Model.

Paris & Wood [15] succesfully solved their model equation (2 1-7), by using Airy functions and the Stokes phenomenon They found the imaginary part of the eigenvalue to be

$$\mathrm{Im}\lambda \sim -\frac{2h^2}{e} exp\left(\frac{-4h^3}{3\varepsilon}\right), \qquad \varepsilon \to 0^+ \qquad (2\ 4\text{-}1)$$

Brazel, Lawless and Wood [3], solved the case for the delta function potential with the term resulting from the bending εx replaced by εx^2, by using Weber's solutions of the parabolic cylinder equation They found the imaginary part of the eigenvalue to be

$$\mathrm{Im}\lambda \sim -2h^2 exp\left(\frac{-\pi h^2}{(2\varepsilon^{1/2})}\right), \qquad \varepsilon \to 0^+ \qquad (2\ 4\text{-}2)$$

Liu and Wood [11], solved the case for the delta function potential with this time the term resulting from the bending εx replaced by εx^n, $n \geq 3$ There were no special functions available for them to use, so they had to rely on other methods They first identified the WKB approximate solution for large x, which satisfied the outgoing wave condition and matched this to the Airy function approximate solution, valid in the neighbourhood of the turning point $x = (-\lambda/\varepsilon)^{1/n}$ nearest to the positive real axis They then substituted this into the boundary condition at the origin, which led to an eigenvalue relation and thus a general formula for $\mathrm{Im}(\lambda)$ given by

$$\mathrm{Im}\lambda \sim -2h^2 exp\left\{\frac{-2h^{(n+2)/(n)}}{\varepsilon^{1/n}}S(n)\right\}, \qquad \varepsilon \to 0^+, \qquad (2\ 4\text{-}3)$$

where

$$S(n) = \frac{\Gamma(1/n + 1)\Gamma(3/2)}{\Gamma(3/2 + 1/n)}, \qquad n \in Z^+ \qquad (2\ 4\text{-}4)$$

Chapter 3

A Model For Fibres With W-Shaped Profiles.

In Chapter 2, we saw how Burzlaff & Wood [4] put forward a more realistic profile for the slightly bent, monomode fibre, (2 3-10)-(2 3-11) However, like Paris & Wood's model, it is still idealistic in nature, picked because it can also be solved explicitly, although technically more difficult, using special functions whose asymptotic properties are well known In industry and in the production of fibres, things are seldom ideal and frequently flaws occur

In this chapter we will concentrate on one particular method used in the production of monomode optical fibres We will describe in detail how the fibre is formed and comment on a flaw that frequently occurs in it We will put forward a profile for this fibre and mathematically incorporate it, into a model similar in nature to Burzlaff & Wood's model Then, we will obtain boundary conditions, particular to our profile, both at the origin and at infinity, in much the same way as Burzlaff & Wood obtained theirs Finally, we will rescale our model, into a form, which will enable us to calculate the radiation loss, using a method proposed by Hu & Cheng in [6]

3.1 Our Model Problem.

There are various methods used in industry, in the production of optical fibres One such method, known as C V D (Chemical Vapour Decomposition) is frequently used and is described as follows A tube of commercial fused silica of, for example, 1m in length and with a 20mm inner diameter, is rotated on a lathe and is heated externally by an oxyhydrogen burner The hot zone is only a few centimetres long and can be shifted along the tube The vapourised source material is passed through the tube together with oxygen in a proportion appropriate to the type of glass to be produced In the hot zone, at about 1600^0C, it oxidises and then is deposited on the inner surface of the tube as a thin layer of oxide By shifting the hot zone back and forth many times and simultaneously altering the gaseous mixture, the desired refractive-index profile, is produced through the accumulation of many layers of varying composition After the deposition process, the tube is heated further, until at about 2000^0C its softening temperature is reached, and the tube contracts and finally collapses under the influence of surface tension, to form a solid glass rod of roughly 10mm in diameter, called a preform It contains in its interior the refractive index profile of the fibre-to-be Now, frequently, a flaw occurs in the process, known in the industry as 'Burnout' It results in a refractive index dip along the centre of the preform, (cf Fig 3 1) described by the profile

$$\mathrm{n}^2(\mathrm{R}) = \begin{cases} t + \frac{2h}{p}\ [\mathrm{R}^m + \beta(1-\mathrm{R})^\sigma] \ = \mathrm{n}_{\mathrm{co}}^2, & \text{when } \mathrm{R} \le 1 \\ t = \mathrm{n}_{\mathrm{cl}}^2, & \text{when } \mathrm{R} \ge 1 \end{cases} \qquad (3\ 1\text{-}1)$$

where,

$$\begin{aligned} \mathrm{R} &= x/\mathrm{a}, \text{the normalised radial coordinate}, \\ \mathrm{n_{co}} &= \text{Refractive index in the core}, \\ \mathrm{n_{cl}} &= \text{Refractive index in the cladding}, \\ 2\mathrm{h}/\mathrm{p} &= \text{Height of the profile, where } \mathrm{h} \ge 0, \\ \mathrm{a} = \mathrm{p}/2 &= \text{Core radius}, \end{aligned}$$

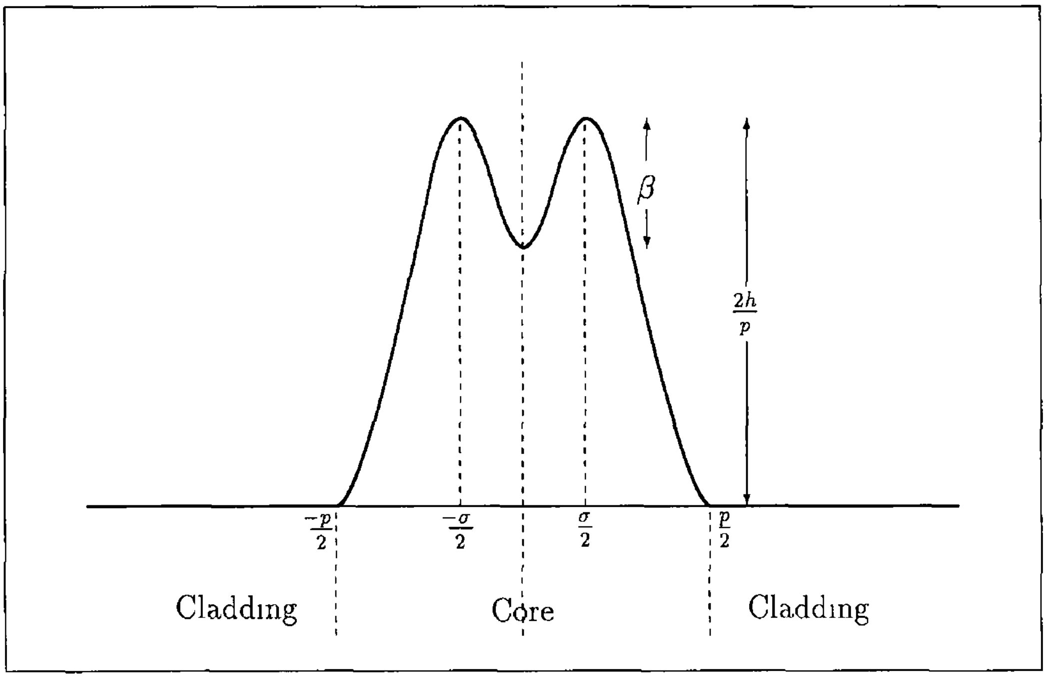

Figure 3 1 Our realistic W-shaped optical fibre profile

$$\beta = \text{Scaled central dip,}$$
$$\sigma = \text{Width of the central dip,}$$
$$\mathrm{m} = \text{Grading of the profile}$$

Here, the parameters m, β and σ have the following constraints imposed on them,

$$1\ 5 \leq \mathrm{m} \leq \infty, \qquad 0 \leq \beta \leq 1, \qquad 4 \leq \sigma \leq \infty \tag{3 1-2}$$

From (3 1-1), we see that we have a power law graded index profile, with a refractive index profile centered along the axis of the fibre

3.2 Comparison of our Model Problem to Burzlaff and Wood's Model.

We will now apply (3 1-1) to Kath & Kriegsmann's paper [9], to develop an expression for the refractive index in the core relative to the cladding This will enable us to convert our profile (3 1-1) into a model similar in nature to Burzlaff & Wood's So, applying (3 1-1) to (1 1-2), we get

$$\mathrm{n}^2(\mathrm{R}) = \frac{\mathrm{n}_{\mathrm{co}}^2}{\mathrm{n}_{\mathrm{cl}}^2} = \frac{t + \frac{2h}{p}\left[\mathrm{R}^m + \beta(1-\mathrm{R})^\sigma\right]}{\mathrm{t}}, \tag{3 2-1}$$

$\Rightarrow$

$$\mathrm{n}^2(\mathrm{R}) = 1 + \frac{2h}{pt}\left[\mathrm{R}^m + \beta(1-\mathrm{R})^\sigma\right] \tag{3 2-2}$$

Comparing (3 2-2) with (1 1-2), reveals that,

$$f(\xi,\eta) = f(\mathrm{R}) = \frac{2hk^2}{pt}\left[\mathrm{R}^m + \beta(1-\mathrm{R})^\sigma\right], \tag{3 2-3}$$

which gives us our expression for the refractive index in the core, relative to the cladding

Now, comparing our profile (Fig 3 1), to Burzlaff & Wood's (Fig 2 2), one can see that the core radius of both is $p/2$ Therefore, our expression for the normalised radial coordinate becomes,

$$\mathrm{R} = \frac{x}{\mathrm{a}} = \frac{2x}{p} \tag{3 2-4}$$

and

$$f(\mathrm{R}) = f\left(\frac{2x}{p}\right) = \frac{2hk^2}{pt}\left[\left(\frac{2x}{p}\right)^m + \beta\left(1-\frac{2x}{p}\right)^\sigma\right], \tag{3 2-5}$$

which we will now denote by $D(x)$

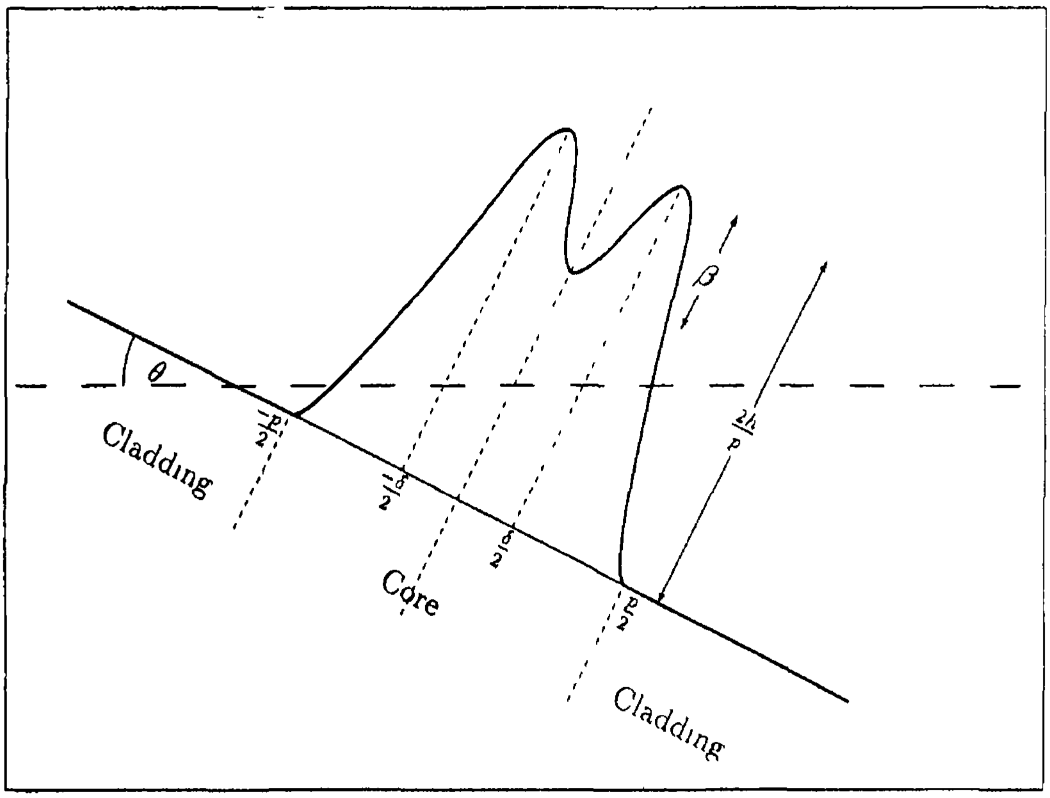

Figure 3 2 Our profile for a bent fibre

Our model (cf Fig 3 2)[1], for simplicity symmetrised with respect to left-right reflection, is given by

$$- y''(x) + v(x)y(x) = \lambda\, y(x), \qquad x \in (-\infty, \infty), \tag{3 2-6}$$

where,

$$v(x) = \begin{cases} -\varepsilon \mid x \mid, & \text{when } \mid x \mid > \frac{p}{2} \\ -D(\mid x \mid) - \varepsilon \mid x \mid, & \text{when } \mid x \mid \leq \frac{p}{2} \end{cases} \tag{3 2-7}$$

Now, like in Burzlaff & Wood's model [4], we are still only interested in the behaviour of our solutions near a turning point situated well into the cladding, that is where $D(\mid x \mid) = 0$ Because of this, we will impose the restriction on $y(x)$ that it must have a controlling behaviour of the form $e^{ip(x)}$ where $p(x)$ is a positive function of

[1] θ is the rotation of the fibre which removes the torsion

x, as $x \to +\infty$ A comparison of (3 2-7), with (2 3-4), shows that $D(|x|)$ replaces the constant function $-2h/p$ We shall now show that this causes the two models to also differ in the expression of the boundary condition at the origin

As before, (2 3-5), we integrate across the core from $-p/2$ to $+p/2$, and take the limit as $p \to 0$ Following the method of Burzlaff & Wood we impose the symmetry conditions $y(x) = y(-x)$ and $y'(-x) = -y'(x)$, confining our model to the half axis We get the limit as $p \to 0$ of,

$$
\begin{aligned}
-\int_{-p/2}^{+p/2} y''(x)\,dx \quad &- \int_{-p/2}^{+p/2} D(|x|)\, y(x)dx \\
&- \varepsilon \int_{-p/2}^{+p/2} x\, y(x)dx = \lambda \int_{-p/2}^{+p/2} y(x)dx,
\end{aligned} \tag{3 2-8}
$$

giving us,

$$
-y'(x)\Big|_{0-}^{0+} - \lim_{p \to 0} \int_{-p/2}^{+p/2} D(|x|)\, y(x)dx = 0 \tag{3 2-9}
$$

Obviously, the integral in (3 2-9), poses a problem To solve it we will use a limiting form of the delta function, to show that the integral in (3 2-9) may be replaced by the value of y at the origin, multiplied by a constant factor containing the shape parameters, β, σ and m, of the core refractive index profile The theorem which enables us to do so, can be found in (p 110, [18]) Because of its importance to this project, we will state it below

3.3 Limit Theorem for the Delta Function.

Let $f(x) = f(x_1, \quad , x_n)$ be a nonnegative locally integrable function on $\Re_n$ for which $\int_{\Re_n} f(x)dx = 1$ With $\alpha < 0$ define

$$
f_\alpha(x) = \frac{1}{\alpha^n} f\left(\frac{x}{\alpha}\right) = \frac{1}{\alpha^n} f\left(\frac{x_1}{\alpha}, \quad , \frac{x_n}{\alpha}\right), \tag{3 3-1}
$$

then $\{f_\alpha(x)\}$ is a delta family as $\alpha \to 0$ [and, setting $\alpha = 1/k$, the sequence $\{s_k(x) = k^n f(k\,x_1, \quad , k\,x_n)\}$ is a delta sequence as $k \to \infty$] The substitution

$y = x/a$ yields these three properties

$$\text{(a)} \quad \int_{\Re_n} f_\alpha(x)dx = 1, \tag{3 3-2}$$

$$\text{(b)} \quad \lim_{\alpha\to 0} \int_{|x|>A} f_\alpha(x)dx = 0 \quad \text{for each A} > 0, \tag{3 3-3}$$

$$\text{(c)} \quad \lim_{\alpha\to 0} \int_{|x|<A} f_\alpha(x)dx = 1 \quad \text{for each A} > 0 \tag{3 3-4}$$

so that for small positive α, $f_\alpha(x)$ is highly peaked about $x = 0$ in such a way that the total strength of this distributed source is unity, with most of it near the origin Also, from [18], we can take the property that,

$$\lim_{\alpha\to\alpha_0} \int_{\Re_n} f_\alpha(x)\phi(x)dx = \phi(0), \tag{3 3-5}$$

for each ϕ in $C_0^\infty(\Re_n)$, where as before $\{f_\alpha\}$ is an n–dimensional delta family (as $\alpha \to \alpha_0$)

3.4 The Delta Function Limit of our Model.

In our case $n = 1$, as we are dealing with a one-dimensional model Thus, for the theorem to hold

$$\int_{-p/2}^{+p/2} D(\mid x \mid)\, dx \tag{3 4-1}$$

must be equal to unity Clearly, this is unlikely to be the case, but we will evaluate the integral, to obtain a scaling factor H

$$\begin{aligned}
\int_{-p/2}^{+p/2} D(\mid x \mid)\, dx &= \frac{2\,h\,k^2}{p\,t}\int_{-p/2}^{+p/2}\left[\left(\frac{2\mid x\mid}{p}\right)^m + \beta\left(1-\frac{2\mid x\mid}{p}\right)^\sigma\right]dx \\
&= \frac{2^{m+1}\ h\ k^2}{p^{m+1}\ t}\left[\int_{-p/2}^{0}(-x)^m\ dx + \int_0^{+p/2}(x)^m\ dx\right] \\
&\quad + \frac{2\,h\,k^2\,\beta}{p\,t}\left[\int_{-p/2}^{0}\left(1+\frac{2x}{p}\right)^\sigma dx + \int_0^{+p/2}\left(1-\frac{2x}{p}\right)^\sigma dx\right] \\
&= \frac{2^{m+1}\,h\,k^2}{p^{m+1}\,t}\left[\frac{2(p/2)^{m+1}}{m+1}\right] + \frac{h\,k^2\,\beta}{t}\left[\frac{2}{\sigma+1}\right] \\
&= \frac{2hk^2}{t}\left[\frac{1}{m+1}+\frac{\beta}{\sigma+1}\right] = H \neq 1
\end{aligned} \tag{3 4-2}$$

Now, letting a function $G(x)$, be a scaled version of $D(x)$, defined by

$$G(x) \;=\frac{D(x)}{\mathrm{H}}, \tag{3 4-3}$$

we get the desired property,

$$\int_{-p/2}^{+p/2} G(x)dx = 1 \tag{3 4-4}$$

So from (3 4-2) and (3 4-3) we get,

$$G(x) = \frac{1}{\left[\frac{1}{m+1}+\frac{\beta}{\sigma+1}\right]}\left[\frac{2^m| \, x \, |^m}{p^{m+1}}+\frac{\beta}{p}\left(1-\frac{2| \, x \, |}{p}\right)^{\sigma}\right] \tag{3 4-5}$$

Thus, to apply the theorem, we must choose an α and rewrite $G(x)$ into the form

$$G_\alpha(x) = \frac{1}{\alpha}G\left(\frac{x}{\alpha}\right) \tag{3 4-6}$$

Choosing $\alpha = p/2$, we get

$$G_{p/2}(x) = \frac{1}{\left[\frac{1}{m+1}+\frac{\beta}{\sigma+1}\right]}\frac{2}{p}\left[\frac{1}{2}\left(\frac{2\mid x\mid}{p}\right)^m+\frac{\beta}{2}\left(1-\frac{2\mid x\mid}{p}\right)^{\sigma}\right], \tag{3 4-7}$$

so as $p/2 \rightarrow 0$,

$$G_{p/2}(x) \longrightarrow \frac{1}{\left[\frac{1}{m+1}+\frac{\beta}{\sigma+1}\right]}\delta(x), \tag{3 4-8}$$

where $\delta(x)$, is the delta function

Hence (3 2-9), is replaced by

$$-\mathrm{y}'(x)\Big|_{0^-}^{0^+} - \frac{1}{\left[\frac{1}{m+1}+\frac{\beta}{\sigma+1}\right]}\int_{0^-}^{0^+}\delta(x)\mathrm{y}(x)dx \;=\; 0 \tag{3 4-9}$$

and we get, using (3 3-5)

$$-2y'(0) - \frac{1}{\left[\frac{1}{m+1}+\frac{\beta}{\sigma+1}\right]}\,y(0) \;=\; 0, \tag{3 4-10}$$

$\Rightarrow$

$$y'(0) + \frac{1}{2}\left[\frac{1}{\left[\frac{1}{m+1} + \frac{\beta}{\sigma+1}\right]}\right] y(0) = 0 \qquad (3\ 4\text{-}11)$$

Therefore, we are left with the following non–self–adjoint eigenvalue problem,

$$y''(x) + [\lambda + \varepsilon x]\, y(x) = 0, \qquad (3\ 4\text{-}12)$$

with boundary condition,

$$y'(0) + \left[\frac{1}{2\left[\frac{1}{m+1} + \frac{\beta}{\sigma+1}\right]}\right] y(0) = 0, \qquad (3\ 4\text{-}13)$$

at the origin, where $x \in (0, \infty)$ and $y(x)$, has controlling behaviour of the form $e^{ip(x)}$, where $p(x)$ is a positive function of x, as $x \to +\infty$ Now, it is obvious from our boundary condition at the origin (3 4-13) that the physical constants, m, β and σ, on which our profile depends, are present Therefore, the information on the shape of the refractive index profile in the core is included in our problem, via the boundary condition at the origin

3.5 Another Form of the Equation.

Finally in this chapter, we will convert (3 4-12)-(3 4-13), into the form required by Hu & Cheng's method [6] So, substituting $x = \varepsilon^q t$, into (3 4-12), we get,

$$\varepsilon^{-2q}\frac{d^2 y}{dt^2} + \left[\lambda + \varepsilon^{q+1} t\right] y = 0 \qquad (3\ 5\text{-}1)$$

Here we used the fact that

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \varepsilon^{-q}\,\frac{dy}{dt} \qquad (3\ 5\text{-}2)$$

and

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\varepsilon^{-q}\,\frac{dy}{dt}\right) = \varepsilon^{-2q}\,\frac{d^2 y}{dt^2} \qquad (3\ 5\text{-}3)$$

Now, by letting $q = -1$, we get the form required by Hu & Cheng [6],

$$\varepsilon^2 \frac{d^2 y}{dt^2} + [\lambda + t]\, y = 0, \tag{3 5-4}$$

where $\lambda \in \mathbb{C}$ and $t \in (0, \infty)$

Also, (3 4-13) is replaced by

$$\varepsilon\ y'(0)\ +\ \left[\frac{1}{2\left[\frac{1}{m+1} + \frac{\beta}{\sigma+1}\right]}\right] y(0)\ =\ 0 \tag{3 5-5}$$

So, (3 4-12)-(3 4-13) becomes

$$\varepsilon^2 y''(t) + [\lambda + t]\, y(t) = 0, \tag{3 5-6}$$

$$\varepsilon y'(0) + \left[\frac{1}{2\left[\frac{1}{m+1} + \frac{\beta}{\sigma+1}\right]}\right] y(0) = 0, \tag{3 5-7}$$

where $t \in (0, \infty)$ and y(t) has controlling behaviour of the form $e^{ip(t)}$, where $p(t)$ is a positive function at t, as $t \to +\infty$

Comparing the above problem, with adiabatic invariance problems, or reflection coefficient problems (see [1],[7],[8],[16],[13],etc) one can clearly see that they share not only similar equations, but also similar methods, to compute the radiation loss, as well as the reflection coefficient

Chapter 4

Review of Results of Gingold, Hu and Cheng.

From Chapter 3, we are left with (3 5-6), a non-self-adjoint Sturm-Liouville problem, with boundary conditions (3 5-7) at the origin and behaviour $e^{ip(t)}$, $p(t) > 0$ as $t \to +\infty$ Now, to solve (3 5-6) we could use special functions (that is 'Airy' functions), whose Poincaré asymptotic properties are well known and whose exponential improved expansions have recently been obtained Information gathered from these exponentially small corrections is absolutely essential for us to calculate the transcendentally small quantity Im(λ), that is the radiation loss

However, this method has already been employed by various authors ([14], [19],[15],[11]) As a result, we will use a relatively new method proposed by Jishan Hu & Wing-Cheong Cheng in their 1990 paper [6] In this paper, the authors make use of the work developed by Harry Gingold [5] to obtain approximate asymptotic solutions to problems similar in nature to (4 2-1) Having obtained these solutions they then use a method (developed initially to solve reflection coefficient problems, by Gingold & Hu [7]) to obtain an expression for the transcendentally small radiation loss

4.1 Review of Asymptotic Results for Differential Equations.

Before we move on and obtain solutions to our particular problem (3 5-6)-(3 5-7), the question must be asked "why not apply more well-known methods to our problem?" We know that in the mathematical sciences, there exists a voluminous amount of literature, dealing with asymptotic formulas for the approximation of solutions, to equations of the form

$$y''(t) = \varphi(t)y(t) \tag{4 1-1}$$

The 'Liouville-Green' approximation, the basis of the 'WKB Approximation', seems to be the earliest and obvious example However, although this method is valid at an irregular singularity of (4 1-1), it fails at a regular singularity and also in the very important case of a turning point

Another method which could be used, although not strictly speaking an asymptotic formula is the Frobenius method Unfortunately, although it is applicable in the neighbourhood of a regular singular point, it fails in the neighbourhood of an irregular singularity Various other asymptotic formulas improve their validity as the variable tends to infinity, but become invalid at a finite point It is clear, therefore, that although there are various asymptotic methods available to solve equations of type (4 1-1), they all fail at certain points on the infinite interval

Clearly, what is needed is an asymptotic formula which is valid throughout the entire infinite interval Gingold's formulas fulfill this need He proved them to be valid in a half neighbourhood of a point t_o, irrespective of whether t_o is a regular or an irregular singular point He also showed (except for some exceptional cases) that his formulas are valid at a turning point Finally, he provided examples whereby ordinary differential equations were taken on an infinite interval, including singularities at the endpoints and thus provided a uniformly valid approximation on the entire infinite interval It is therefore only natural to label these formulas 'invariant' Here, Gingold means 'invariant' in the sense that they are valid all the way up to a turning point

4.2 Outline of Derivation and Results of Gingold.

Gingold's formulas will, as we have already mentioned allow us to obtain two linearly independent asymptotic solutions to (4 1-1) and their respective derivatives But how did Gingold obtain these approximate solutions? In this section, we will give a brief outline of the method he used First however, we will need to define the following functions and assume that they adhere to Convention 2 1 & Assumption 2 2, p 320 [5]

$$
\begin{aligned}
L(t) &= \frac{1}{4}\frac{\varphi'(t)}{[\varphi(t)]^{3/2}}, && (4\ 2\text{-}1)\\
\Theta(t) &= ln\left[\frac{1-\imath L(t)}{1+\imath L(t)}\right]^{1/4}, && (4\ 2\text{-}2)\\
J(t) &= \sqrt{\varphi(t)+\left(\frac{1}{4}\frac{\varphi'(t)}{\varphi(t)}\right)^2}, && (4\ 2\text{-}3)\\
r(t) &= \frac{\imath}{2}\frac{L'(t)}{1+L^2(t)}, && (4\ 2\text{-}4)\\
e(t,\tau) &= exp\left[2\int_\tau^t J(s)ds\right] && (4\ 2\text{-}5)
\end{aligned}
$$

Gingold begins by rewriting (4 1-1) into its companion matrix differential system

$$
\mathbf{Y}' = \begin{bmatrix} 0 & 1 \\ \varphi(t) & 0 \end{bmatrix}\mathbf{Y}, \qquad \mathbf{Y} = \begin{pmatrix} y \\ y' \end{pmatrix} \qquad (4\ 2\text{-}6)
$$

He then performs two successive linear transformations,

$$
\mathbf{Y} = W_1\mathbf{Y}_1 \qquad \text{and} \qquad \mathbf{Y}_1 = W_2\mathbf{Y}_2, \qquad (4\ 2\text{-}7)
$$

where

$$
W_1 = \begin{bmatrix} [\varphi(t)]^{-1/4} & [\varphi(t)]^{-1/4} \\ [\varphi(t)]^{+1/4} & [\varphi(t)]^{+1/4} \end{bmatrix}, \qquad W_2 = \frac{1}{2}\begin{bmatrix} \vartheta & -\psi \\ \imath\psi & -\imath\vartheta \end{bmatrix},
$$

$$\begin{aligned} \vartheta &= m + m^{-1} = 2\cosh\Theta(t), \\ \psi &= m - m^{-1} = 2\sinh\Theta(t) \end{aligned} \qquad (4\ 2\text{-}8)$$

and

$$m = \left[\frac{1 - \imath L(t)}{1 + \imath L(t)}\right]^{1/4} \qquad (4\ 2\text{-}9)$$

Here $L(t)$ and $\Theta(t)$ are defined by (4 2-1) and (4 2-2) respectively This enabled him, to transform (4 1-1) into a form amenable to a method of diagonalisation Then, using various assumptions, conventions and lemmas [1], with proofs supplied, he was able to obtain a fundamental solution set of (4 1-1) as $t \to +\infty$, given by [2]

$$\mathbf{Y}(t) = W_1(t)W_2(t)\,(I + P(t))\,Z(t), \qquad (4\ 2\text{-}10)$$

where I is the identity matrix and

$$Z(t) = \begin{bmatrix} exp\left\{+\int^t J(s)ds\right\} & 0 \\ 0 & exp\left\{-\int^t J(s)ds\right\} \end{bmatrix} \qquad (4\ 2\text{-}11)$$

Thus, using (4 2-10), Gingold was able to obtain a fundamental solution set of (4 1-1), that is two linearly independent solutions and their respective derivitives, which we will now state below

$$\begin{aligned} y_1(t) &= [\psi(t)]^{-1/4}\left\{[cosh\Theta(t) + \imath sinh\Theta(t)]\,(1 + p_{11})\right. \\ &\quad \left.- \imath\,[cosh\Theta(t) - \imath sinh\Theta(t)]\,p_{21}\right\} exp\left\{+\int^t J(s)ds\right\}, \qquad (4\ 2\text{-}12) \\ y_2(t) &= [\psi(t)]^{-1/4}\left\{[cosh\Theta(t) + \imath sinh\Theta(t)]\,p_{12}\right. \\ &\quad \left.-\imath\,[cosh\Theta(t) - \imath sinh\Theta(t)]\,(1 + p_{22})\right\} exp\left\{-\int^t J(s)ds\right\}, \qquad (4\ 2\text{-}13) \\ y_1'(t) &= [\psi(t)]^{+1/4}\left\{[cosh\Theta(t) - \imath sinh\Theta(t)]\,(1 + p_{11})\right. \\ &\quad \left.+ \imath\,[cosh\Theta(t) + \imath sinh\Theta(t)]\,p_{21}\right\} exp\left\{+\int^t J(s)ds\right\}, \qquad (4\ 2\text{-}14) \end{aligned}$$

[1] see lemma 2 1,[7]

[2] (I + P(t)) is a continuously invertible 2 x 2 matrix function

$$y_2'(t) = [\psi(t)]^{+1/4}\{[cosh\Theta(t) - \imath sinh\Theta(t)]\,p_{12}$$
$$+\imath\,[cosh\Theta(t) + \imath sinh\Theta(t)]\,(1+p_{22})\}\,exp\left\{-\int^t J(s)ds\right\} \quad (4\ 2\text{-}15)$$

Now, the entries of $P(t)$, $p_{\jmath,k}$, $(\jmath, k = 1, 2)$, satisfy a Volterra integral equation, expressed in [7] Under certain conditions they form convergent series for any $t \in [a, b]$ given by [3]

$$p_{11}(t,\alpha_{11},\alpha_{21}) = \sum_{m=0}^{+\infty}\int_{\alpha_{11}}^{t} r(\hat{t}_0)d\hat{t}_0\int_{\alpha_{21}}^{\hat{t}_0} r(t_0)e(t_0,\hat{t}_0)dt_0$$
$$\prod_{n=1}^{m}\int_{\alpha_{11}}^{t_{n-1}} r(\hat{t}_n)d\hat{t}_n\int_{\alpha_{21}}^{\hat{t}_n} r(t_n)e(t_n,\hat{t}_n)dt_n, \quad (4\ 2\text{-}16)$$
$$p_{22}(t,\alpha_{12},\alpha_{22}) = \sum_{m=0}^{+\infty}\int_{\alpha_{22}}^{t} r(\hat{t}_0)d\hat{t}_0\int_{\alpha_{12}}^{t_0} r(t_0)e(\hat{t}_0,t_0)d\hat{t}_0$$
$$\prod_{n=1}^{m}\int_{\alpha_{22}}^{t_{n-1}} r(\hat{t}_n)d\hat{t}_n\int_{\alpha_{12}}^{t_n} r(t_n)e(\hat{t}_n,t_n)dt_n, \quad (4\ 2\text{-}17)$$
$$p_{12}(t,\alpha_{12},\alpha_{22}) = \sum_{m=0}^{+\infty}\int_{\alpha_{12}}^{t} r(\hat{t}_0)e(t,\hat{t}_0)d\hat{t}_0$$
$$\prod_{n=1}^{m}\int_{\alpha_{22}}^{t_{n-1}} r(\hat{t}_n)d\hat{t}_n\int_{\alpha_{12}}^{t_n} r(t_n)e(\hat{t}_n,t_n)dt_n, \quad (4\ 2\text{-}18)$$
$$p_{21}(t,\alpha_{11},\alpha_{21}) = \sum_{m=0}^{+\infty}\int_{\alpha_{21}}^{t} r(\hat{t}_0)e(\hat{t}_0,t)d\hat{t}_0$$
$$\prod_{n=1}^{m}\int_{\alpha_{11}}^{t_{n-1}} r(\hat{t}_n)d\hat{t}_n\int_{\alpha_{21}}^{\hat{t}_n} r(t_n)e(t_n,\hat{t}_n)dt_n, \quad (4\ 2\text{-}19)$$

where $\alpha_{\jmath k}$, $\jmath, k = 1, 2$, are arbitrary constants in [a,b]

4.3 Hu & Cheng's Application of Gingold's Results.

In section 4 2, we showed how Gingold obtained approximate asymptotic solutions to equations of the form (4 1-1) and their respective derivatives We will now show how Gingold's formulas can be used for more arbitrary functions and not just for functions with an eigenvalue dependence such as ours If we replace $\lambda + t$, in (3 5-6), with $Q_+(t)$ we get

$$\varepsilon^2\, y''(t) + [Q_+(t)]\, y(t) = 0 \quad (4\ 3\text{-}1)$$

[3] see [7], Lemma 2 1

Here we assume that $Q_+(t)$ adheres to the following conditions

(1) $Q_+(t) \in C^\infty([a\,,b])$, with $0 < a < b < \infty$,

(thus making the function continuous and infinitely differentiable on [a, b]),

(2) $Q_+(t) \neq 0 \;\forall\; t \in (a, b)$,

(thus ensuring (4 3 − 1) has no turning point on (a, b)),

(3) $\left|\dfrac{Q'_+(t)^2}{Q_+(t)^3}\right| \leq M\,,\;\forall\; t \in [a\,,b]$,

(thus ensuring that $L^2(t) \neq -1\,,\forall\; t \in [a, b]$),

(4) $\displaystyle\int_a^b \left|\left(\frac{Q'_+(t)}{Q_+^{3/2}(t)}\right)'\right| dt \leq \infty$,

(thus ensuring the existence of $L^2(t)$ everywhere on the real line) (4 3-2)

We can obtain asymptotically approximate solutions for a wider class of potential functions Assumption (3) is imposed to ensure an induced turning point never occurs That is, there exists no $t \in [a\,,b]$ such that $J(t) \neq 0$ If such a t did exist, it would render our solutions trivial By making these assumptions, we are able to obtain, using Gingold's formulas, two linearly independent solutions of (4 3-1) and their respective derivatives, that is (4 2-12)-(4 2-15), with $\varphi(t)$ replaced with $-Q_+(t)/\varepsilon^2$, for a general function Q_+

Assumption (4) ensures that the difference between the controlling behaviours of our solutions and the controlling behaviours obtained by a Liouville-Green approximation, that is

$$\int^t \sqrt{\frac{-Q_+(s)}{\varepsilon^2} + \left(\frac{1}{4}\frac{Q'_+(s)}{Q_+(s)}\right)^2} ds - \int^t \sqrt{\frac{-Q_+(s)}{\varepsilon^2}} ds, \qquad (4\ 3\text{-}3)$$

is uniformly bounded So we can extract from our solutions their WKB approximations and their respective derivatives given by

$$y(t) \sim c_1 \left[-Q_+(t)\right]^{-1/4} exp\left\{+\int^t J(s)ds\right\}$$

$$+ c_2 [-Q_+(t)]^{-1/4} exp\left\{- \int^t J(s)ds\right\}, \qquad (4\ 3\text{-}4)$$

$$y'(t) \sim c_1 [-Q_+(t)]^{+1/4} exp\left\{+ \int^t J(s)ds\right\}$$

$$+ c_2 [-Q_+(t)]^{+1/4} exp\left\{- \int^t J(s)ds\right\}, \qquad (4\ 3\text{-}5)$$

where

$$J(t) = \sqrt{\frac{-(Q_+(t))}{\varepsilon^2} + \left(\frac{1}{4}\frac{Q'_+(t)}{Q_+(t)}\right)^2} \qquad (4\ 3\text{-}6)$$

4.4 Gingold's Formulas Applied to our Model Problem.

Before we move on to obtain an expression for the radiation loss in the next chapter, we will first apply Gingold's formulas to our particular problem As it happens we do not require approximate asymptotic solutions to our problem (3 5-6)-(3 5-7) as our local solution is our global solution and asymptotic matching is not required However, we will obtain its solutions in this section, purely as an example of how Gingold's formulas can be applied to any function Q_+ which obeys conditions (4 3-2)

So, by rewriting (3 5-6) into the form (4 1-1), we get

$$y''(t) + \left[\frac{(\lambda + t)}{\varepsilon^2}\right] y(t) = 0, \qquad t \in (0, +\infty) \qquad (4\ 4\text{-}1)$$

with boundary condition

$$\varepsilon y'(0) + \left[\frac{1}{2\left[\frac{1}{m+1} + \frac{\beta}{\sigma+1}\right]}\right] y(0) = 0, \qquad (4\ 4\text{-}2)$$

where λ is a complex eigenvalue, $0 < \varepsilon < 1$ and m, β, σ are defined by (3 1-1) Comparing (4 1-1), with (4 4-1), we find that

$$\varphi(t) = \frac{-(\lambda + t)}{\varepsilon^2}, \qquad (4\ 4\text{-}3)$$

so substituting (4 4-3), into (4 2-12)-(4 2-15), gives us

$$
\begin{aligned}
y_1(t) &= \left[\frac{-(\lambda+t)}{\varepsilon^2}\right]^{-1/4} \{[cosh\Theta(t) + \imath sinh\Theta(t)]\,(1+p_{11}) \\
&\quad - \imath\,[cosh\Theta(t) - \imath sinh\Theta(t)]\,p_{21}\}\,exp\left\{+\int^t J(s)ds\right\}, \qquad (4\ 4\text{-}4) \\
y_2(t) &= \left[\frac{-(\lambda+t)}{\varepsilon^2}\right]^{-1/4} \{[cosh\Theta(t) + \imath sinh\Theta(t)]\,p_{12} \\
&\quad - \imath\,[cosh\Theta(t) - \imath sinh\Theta(t)]\,(1+p_{22})\}\,exp\left\{-\int^t J(s)ds\right\}, \qquad (4\ 4\text{-}5) \\
y_1'(t) &= \left[\frac{-(\lambda+t)}{\varepsilon^2}\right]^{+1/4} \{[cosh\Theta(t) - \imath sinh\Theta(t)]\,(1+p_{11}) \\
&\quad + \imath\,[cosh\Theta(t) + \imath sinh\Theta(t)]\,p_{21}\}\,exp\left\{+\int^t J(s)ds\right\}, \qquad (4\ 4\text{-}6) \\
y_2'(t) &= \left[\frac{-(\lambda+t)}{\varepsilon^2}\right]^{+1/4} \{[cosh\Theta(t) - \imath sinh\Theta(t)]\,p_{12} \\
&\quad + \imath\,[cosh\Theta(t) + \imath sinh\Theta(t)]\,(1+p_{22})\}\,exp\left\{-\int^t J(s)ds\right\} \qquad (4\ 4\text{-}7)
\end{aligned}
$$

where

$$
J(t) = \sqrt{\frac{-(\lambda+t)}{\varepsilon^2} + \left(\frac{1}{16(\lambda+t)^2}\right)} \qquad (4\ 4\text{-}8)
$$

We are allowed to do this as $\forall t \in [0, +\infty)$, $\lambda + t \neq 0$ as $\lambda \in \mathbb{C}$ So there are no turning points or induced turning points present, which would render equations (4 4-4)-(4 4-7), nonexistent or trivial That is $\lambda + t$ satisfies conditions (4 3-2)

Now having so far obtained asymptotic approximate solutions to (4 4-1) valid all the way up to a turning point, we can extract from (4 4-4)-(4 4-7) its equivalent WKB approximations We are justified in doing this, as the difference between the controlling behaviours of (4 4-4)-(4 4-7) and the controlling behaviours, obtained by a Liouville-Green approximation, that is

$$
\int^t \sqrt{\frac{-(\lambda+s)}{\varepsilon^2} + \left(\frac{1}{16(\lambda+s)^2}\right)}ds \;-\; \int^t \sqrt{\frac{-(\lambda+s)}{\varepsilon^2}}ds, \qquad (4\ 4\text{-}9)
$$

is uniformly bounded, $\forall t \in [0, \infty]$, $\lambda \in \mathbb{C}$

So, we can express the general solution to (4 4-1) and its derivative, for large positive t, as follows

$$\begin{aligned} y(t) &\sim c_1 \left[\frac{-(\lambda+t)}{\varepsilon^2}\right]^{-1/4} exp\left\{+\int^t J(s)ds\right\} \\ &+ c_2 \left[\frac{-(\lambda+t)}{\varepsilon^2}\right]^{-1/4} exp\left\{-\int^t J(s)ds\right\}, \qquad (4\ 4\text{-}10) \\ y'(t) &\sim c_1 \left[\frac{-(\lambda+t)}{\varepsilon^2}\right]^{+1/4} exp\left\{+\int^t J(s)ds\right\} \\ &+ c_2 \left[\frac{-(\lambda+t)}{\varepsilon^2}\right]^{+1/4} exp\left\{-\int^t J(s)ds\right\}, \qquad (4\ 4\text{-}11) \end{aligned}$$

where

$$J(t) = \sqrt{\frac{-(\lambda+t)}{\varepsilon^2} + \left(\frac{1}{16(\lambda+t)^2}\right)} \qquad (4\ 4\text{-}12)$$

Chapter 5

CALCULATION OF RADIATION LOSS.

We developed in Section 4 4, an asymptotic solution to our problem (3 5-6), as $t \to +\infty$ with boundary condition (3 5-7), at the origin and behaviour $e^{ip(t)}$, $p(t) > 0$ at ∞ In this chapter, we will extend our problem into the complex plane and obtain an expression for the radiation loss Imλ To do this, following Hu and Kruskal [8] we will have to move away from the real axis into the complex t-plane and solve along the nearest level line L_1 on which our critical point $t = -\lambda$ lies First however, we will explain what a level line is, outline what occurs on a level line near a power-type critical point, define what are critical and major critical points and generally attempt to give an understanding to the concepts and ideas used throughout the rest of this chapter

5.1 Critical Level Lines.

To explain what Hu and Kruskal mean by a level line , we must first take a general second-order linear differential equation, say

$$\varepsilon^2 \frac{d^2 y}{dt^2} + Q(t, \lambda) y = 0 \tag{5 1-1}$$

and state its WKB approximate solutions

$$y(t) \sim C_1 \left[\frac{-Q(t,\lambda)}{\varepsilon^2}\right]^{-1/4} exp\left\{+\frac{i}{\varepsilon}\int^t [Q(s,\lambda)]^{1/2}\, ds\right\}$$

$$+ C_2 \left[\frac{-Q(t,\lambda)}{\varepsilon^2}\right]^{-1/4} exp\left\{-\frac{i}{\varepsilon}\int^t [Q(s,\lambda)]^{1/2}\, ds\right\},\ t \to \pm\infty \quad (5\,1\text{-}2)$$

Now, no matter where you move in the complex plane, the magnitude of one of our solutions to (5 1-1), will increase while the magnitude of the other will decrease. That is one of the solutions will become subdominant to the other, depending on where you are in the complex plane. However, if both of our WKB exponentials were of the same order of magnitude, then a level line would occur where both solutions would be of equal importance. So the level lines of (5 1-1),

$$\left|exp\left\{\pm\frac{i}{\varepsilon}\int^z [Q(t,\lambda)]^{1/2}\, dt\right\}\right|, \quad (5\,1\text{-}3)$$

occur where both the WKB exponentials have the same magnitude, that is where

$$\mathrm{Im}\left\{\int^z [Q(t,\lambda)]^{1/2}\, dx\right\} = \mathrm{const} \quad (5\,1\text{-}4)$$

These correspond to anti-Stokes lines in a more conventional treatment. Assume that $Q(z,\lambda)$ is a power-type critical point, whose nature is as yet unknown. Therefore, $Q(z) \sim b(z - z_0)^{2\gamma-2}$, $z \to z_0$, where $b \neq 0$, is a constant and γ is a real number. Then to the leading order, Hu & Kruskal [8] showed that the structure of the level lines of (5 1-1) near $z = z_0$ can be divided topologically into four different classes. (1) If $\gamma < 0$, they consist of rose curves, and the angle of each leaf is $\pi/|\gamma|$. (2) If $\gamma > 0$, they consist of hyperbolic-like curves, and the angle of each leaf is again $\pi/|\gamma|$. (3)(a) If $\gamma = 0$ and $\mathrm{Re}\, b^{1/2} \neq 0$, then they consist of an infinite number of spirals intersecting at $z = z_0$, (b) if $\gamma = 0$ and $\mathrm{Re}\, b^{1/2} = 0$, they consist of an infinite number of circles centred at $z = z_0$.

In our problem, $\gamma = 3/2 > 0$, so (2) above applies. In particular as we shall see, when the constant in (5 1-4) is zero, $\gamma\theta = k\pi$, k = integer, which is a set of half-lines through $z = z_0$. The angle between two consecutive lines is $\pi/|\gamma|$[1]. Finally, in this section, we define a critical point to be a zero or singularity of the general

[1] For further information on (1),(2),(3), see [8]

function $Q(t,\lambda)$ in (5 1-1) We define a 'nearest critical level line' to be that level line on which a critical point first occurs, and we call a critical point a 'major critical point', if it is found on the nearest critical level line

5.2 Hu & Cheng's Method Applied to a General Function $Q(t,\lambda)$.

In this section we will outline briefly the method used by Hu & Cheng [6] in obtaining an expression for the radiation loss from equations of type (5 1-1) with an arbitrary function $Q(t,\lambda)$ From Section 4 3, we showed how Hu & Cheng obtained approximate asymptotic solutions to (4 3-1), (i e (4 4-4)-(4 4-5)), using Gingold's invariant formulas, removing the eigenvalue dependence of $Q(t,\lambda)$ and using $Q_+(t)$ subject to conditions (4 3-2) Combining the outgoing wave solution (i e the solution with positive exponent) with the boundary condition at the origin, left the authors with a well-defined problem given by

$$\begin{cases} \varepsilon^2 y''(t) + Q(t,\lambda)y = 0, & t \in (0,+\infty), \\ \varepsilon y'(0) + hy(0) = 0, & \\ y(t) \sim [Q_+(t)]^{-1/4} exp\left\{+\int_{\bar{t}}^{t} J(s)ds\right\}, & t \to +\infty \end{cases} \tag{5 2-1}$$

where $\bar{t}$ is a sufficiently large real number

Thus the authors were left with the job of obtaining the radiation loss from (5 2-1) To achieve this, they used a method similar to the one used to solve the reflection coefficient problems, studied by Gingold & Hu [7] [2] They began by moving (5 2-1), into the complex t-plane and solved it along the nearest critical level line From (5 1-4), we know that the level lines of (5 2-1) are given by

$$\mathrm{Re}\left\{\int^{t} J(s)ds\right\} = const, \tag{5 2-2}$$

on which the authors assumed there exists at least one critical point of the differential

[2] See [1],[8],[13],[16]

equation where,

$$J(t) = \sqrt{\frac{-Q(t,\lambda)}{\varepsilon^2} + \left(\frac{Q'(t,\lambda)}{4\,Q(t,\lambda)}\right)^2} \tag{5 2-3}$$

Now, as the function $Q(t,\lambda)$ is unknown, the authors made a number of assumptions relating to it They assumed that the eigenvalue λ is known and that $t = t_c$ is a critical point on its nearest level line L_1 They also assumed that near $t = t_c$, $Q(t,\lambda)$ has an asymptotic behaviour of the form

$$Q(t,\lambda) \sim b_c(t - t_c)^{2\gamma_c - 2}, \qquad t \to t_c, \qquad \gamma_c > 0 \tag{5 2-4}$$

With $\gamma_c > 0$, and from our discussion in Section 5 1, the level lines of (5 2-2) are at an angle of $\pi/\,|\gamma|$ from each other and consist of hyperbolic like curves For example, if we take the simplest case and centre our critical point at the origin, that is let $t_c = 0$, $b = 1$ and $\gamma = 3/2$, we get fig 5 0

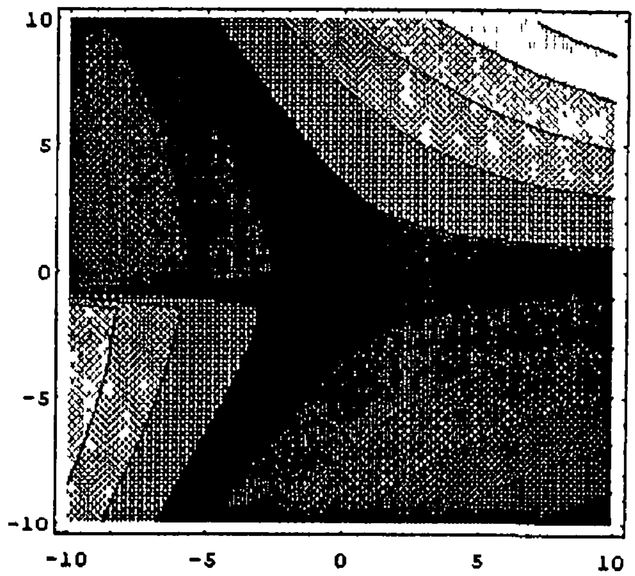

Fig. 5.0. Level curves for (5 2-2)

The authors argued that on L_1 away from the critical point $t = t_c$, the leading behaviour of y can be represented by

$$y(t) \sim [Q(t,\lambda)]^{-1/4}\left[r_1\, exp\left\{+\int_{t_c}^{t} J(s)ds\right\} + r_2\, exp\left\{-\int_{t_c}^{t} J(s)ds\right\}\right], \quad t \to +\infty \qquad (5\ 2\text{-}5)$$

where the values of r_1, r_2 can be determined by the continuation of the behaviour (5 2-1) of y near $+\infty$ So comparing (5 2-5) with (5 2-1) gives

$$\begin{cases} r_1 = exp\left\{-\int_{t_c}^{\bar{t}} J(s)ds\right\}, \\ r_2 = 0 \end{cases} \qquad (5\ 2\text{-}6)$$

The authors then argued that in a neighbourhood of $t = t_c$, the leading term of y satisfies

$$\varepsilon^2 y''(t) + b_c(t - t_c)^{2\gamma_c - 2} y(t) = 0, \qquad (5\ 2\text{-}7)$$

whose local solution can be expressed in terms of Hankel functions, that is

$$y(t) = (t - t_c)^{1/2}\left\{T_1 H^{(1)}_{1/2\gamma_c}\left(\tfrac{b^{1/2}}{\varepsilon\gamma_c}(t-t_c)^{\gamma_c}\right) + T_2 H^{(1)}_{1/2\gamma_c}\left(\tfrac{b^{1/2}}{\varepsilon\gamma_c}(t-t_c)^{\gamma_c}\right)\right\} \qquad (5\ 2\text{-}8)$$

The asymptotic representations of these Hankel functions are given in [8]

So matching their local solution (5 2-8) to their invariant asymptotic solution (5 2-5) allowed them to obtain expressions for T_1 and T_2 given by

$$\begin{cases} T_1 = r_1 \sqrt{\frac{\pi}{2\,\varepsilon\,\gamma_c}}\, e^{\imath \frac{\pi}{4\gamma_c}}\, e^{\imath \frac{\pi}{4}}, \\ T_2 = r_2 \sqrt{\frac{\pi}{2\,\varepsilon\,\gamma_c}}\, e^{-\imath \frac{\pi}{4\gamma_c}}\, e^{-\imath \frac{\pi}{4}} \end{cases} \qquad (5\ 2\text{-}9)$$

However, as they were only interested in the behaviour of their solutions near the origin, they extended the solution (5 2-8), passing through $t = t_c$ from the branch L_1, to the next branch L_2, at the same level in the clockwise direction This is equivalent to a change in the argument of $(t - t_c)^{\gamma_c}$ by $-\pi$ The Hankel transformation formula they used to achieve this can be found in [8]

Hence on L_2, to the leading order, the function y has the form

$$\begin{aligned} y \;=\; & (t-t_c)^{1/2}\, e^{i\frac{\pi}{2\gamma_c}} \left\{ \left[2\,T_1 cos(\pi/2\gamma_c) - T_2\, e^{i\frac{\pi}{2\gamma_c}} \right] H^{(1)}_{1/2\gamma_c}\left(\tfrac{b^{1/2}}{\varepsilon\gamma_c}(t-t_c)^{\gamma_c} \right) \right. \\ & \left. + \left[T_1 e^{-i\frac{\pi}{2\gamma_c}} \right] H^{(2)}_{1/2\gamma_c}\left(\tfrac{b^{1/2}}{\varepsilon\gamma_c}(t-t_c)^{\gamma_c} \right) \right\}. \qquad (5.2\text{-}10) \end{aligned}$$

Now, on L_2 away from the critical point $t = t_c$, Hu & Cheng argued that the function y has the form

$$y \sim Q(t;\lambda)^{-1/4} \left[r_1'\, exp\left\{ + \int_{t_c}^{t} J(s)ds \right\} + r_2'\, exp\left\{ - \int_{t_c}^{t} J(s)ds \right\} \right]. \qquad (5.2\text{-}11)$$

So matching (5.2-11) with (5.2-10) near $t = t_c$, taking into account what occurs to both of them as they pass through the Stokes line $-\pi/\gamma$ gave them

$$\begin{cases} r_1' = T_1 \sqrt{\frac{2\varepsilon\gamma_c}{\pi}}\, e^{-i\frac{\pi}{4\gamma_c}}\, e^{-i\frac{\pi}{4}}, \\ r_2' = e^{i\frac{\pi}{2\gamma_c}} \left[T_1 cos(\pi/2\gamma_c) - T_2\, e^{i\frac{\pi}{2\gamma_c}} \right] \sqrt{\frac{2\varepsilon\gamma_c}{\pi}}\, e^{+i\frac{\pi}{4\gamma_c}}\, e^{+i\frac{\pi}{4}}, \end{cases} \qquad (5.2\text{-}12)$$

thus giving the authors the asymptotic behaviour of their solution near the origin. Finally, using the boundary condition at the origin, (5.2-12) and some mathematical manipulation Hu & Cheng were able to obtain an expression for the radiation loss given by

$$\mathrm{Im}(Q(0,\lambda)) \sim \frac{2\,h^2}{cos(\pi/2\,\gamma_c)}\, cos\,(\pi/\gamma_c \,-\, 2\,\mathrm{Im}\tau)\; e^{-2\mathrm{Re}(\tau)}, \qquad \varepsilon \to 0+, \qquad (5.2\text{-}13)$$

where

$$\tau = \frac{i}{\varepsilon} \int_{t_c}^{0} [Q(s;\lambda)]^{1/2}\, ds. \qquad (5.2\text{-}14)$$

5.3 Derivation of Radiation Loss.

In the previous section we showed how Hu & Cheng in their paper [6] obtained an expression for the radiation loss from (5.2-1), with $Q(t,\lambda)$ an arbitrary function adhering to conditions (4.3-2). In this section, we will apply Hu & Cheng's method to our particular problem. However, unlike Hu & Cheng's method, we do not require

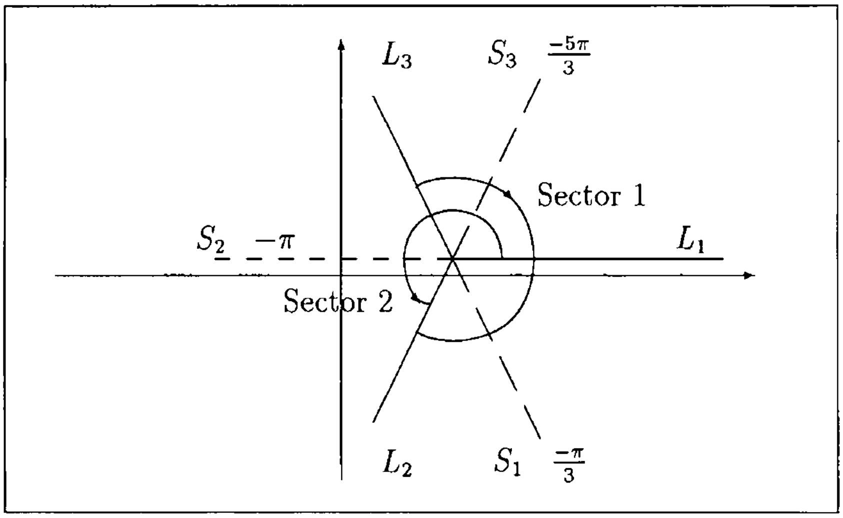

Figure 5 1 Behaviour about critical point t_c

asymptotic matching as $Q(t,\lambda) = \lambda + t$, $\forall t$ and not just for $t \to \infty$ Therefore, the local solution is the global solution, that is

$$y(t) = (t+\lambda)^{1/2} \left\{ T_1\, H^{(1)}_{1/3}\left(\tfrac{2}{3\varepsilon}(\lambda+t)^{3/2}\right) + T_2\, H^{(2)}_{1/3}\left(\tfrac{2}{3\varepsilon}(\lambda+t)^{3/2}\right) \right\}, \qquad (5\ 3\text{-}1)$$

is our global solution to

$$\varepsilon^2 y''(t) + (\lambda + t) y(t) = 0, \qquad t \in (0, +\infty), \qquad (5\ 3\text{-}2)$$

where for the moment the transmission coefficients T_1, T_2 are arbitrary The asymptotic representations for our Hankel functions H^1 and H^2 are

$$\begin{aligned} H^{(1)}_{1/3}\left(\tfrac{2}{3\varepsilon}(\lambda+t)^{3/2}\right) &= \sqrt{\frac{2}{\pi}}\sqrt{\frac{3\varepsilon}{2}}\,(\lambda+t)^{-3/4}\, e^{+i\left(\frac{2}{3\varepsilon}(\lambda+t)^{3/2}\right)}\, e^{-i\frac{5\pi}{12}} \\ &\qquad \left[1 + O\left(\tfrac{3\varepsilon}{2}(\lambda+t)^{-3/2}\right)\right], \qquad (5\ 3\text{-}3) \\ H^{(2)}_{1/3}\left(\tfrac{2}{3\varepsilon}(\lambda+t)^{3/2}\right) &= \sqrt{\frac{2}{\pi}}\sqrt{\frac{3\varepsilon}{2}}\,(\lambda+t)^{-3/4}\, e^{-i\left(\frac{2}{3\varepsilon}(\lambda+t)^{3/2}\right)}\, e^{+i\frac{5\pi}{12}} \\ &\qquad \left[1 + O\left(\tfrac{3\varepsilon}{2}(\lambda+t)^{-3/2}\right)\right] \qquad (5\ 3\text{-}4) \end{aligned}$$

valid as $|\lambda + t| \to \infty$, for $\left|\arg\left(\frac{2}{3\varepsilon}(\lambda+t)^{3/2}\right)\right| < \pi$, (cf Fig 5 1)

Now, as we are dealing with an outgoing wave solution tending to $+\infty$, we must set

$T_2 = 0$ This gives

$$y(t) = (t+\lambda)^{1/2}\left\{T_1 H^{(1)}_{1/3}\left(\tfrac{2}{3\varepsilon}(\lambda+t)^{3/2}\right)\right\}, \qquad t \to \infty, \tag{5 3-5}$$

which has leading asymptotic behaviour,

$$y(t) \sim T_1\,(t+\lambda)^{1/2}\sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{-3/4}\,e^{+i\left(\frac{2}{3\varepsilon}(\lambda+t)^{3/2}\right)}e^{-i\frac{5\pi}{12}}, \qquad t \to \infty, \tag{5 3-6}$$

valid as we have mentioned in sector 1 (see Fig 5 1) We require however, the asymptotic behaviour of our solution near the origin, that is in sector 2 (see Fig 5 1) To obtain this we need to change the argument of $(\lambda + t)^{3/2}$ by $-\pi$ We therefore require the following Hankel connection formula given in [8] [3]

$$H^{(1)}_{1/3}(e^{-i\pi}z) = H^{(1)}_{1/3}(z) + e^{-i\frac{\pi}{3}}H^{(2)}_{1/3}(z), \tag{5 3-7}$$

where

$$z = \frac{2}{3\varepsilon}(\lambda+t)^{3/2} \tag{5 3-8}$$

Hence, (5 3-5) becomes

$$y(t) = T_1\,(t+\lambda)^{1/2}\left\{H^{(1)}_{1/3}(z) + e^{-i\frac{\pi}{3}}H^{(2)}_{1/3}(z)\right\}, \qquad \varepsilon \to 0+ \tag{5 3-9}$$

with leading asymptotic behaviour,

$$y(t) \sim T_1\sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{-1/4}\left\{e^{+i(z)}e^{-i\frac{5\pi}{12}} + e^{-i(z)}\,e^{+i\frac{5\pi}{12}}\,e^{-i\frac{\pi}{3}}\right\}, \qquad \varepsilon \to 0+ \tag{5 3-10}$$

However, as we want the behaviour of our solution on the Stokes line S_2, we need to find the superasymptotic representation, both below and above the Stokes line S_2 and take the average This is the process adopted by Paris & Wood in their 1989 paper [15], subsequently justified by Berry's paper [2], which showed the smooth

[3] taking m to be -1

change of the multiplier, from 0 to 1, with error function dependence, as the Stokes line is crossed This leaves us with the leading asymptotic behaviour

$$\begin{aligned} y(t) &\sim T_1 \sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{-1/4}\, e^{-i\frac{\pi}{2}}\, e^{-i\pi} \left\{ e^{-i(z)} e^{-i\frac{5\pi}{12}} \right\} \\ &\quad + \frac{T_1}{2}\sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{-1/4}\, e^{-i\frac{\pi}{2}}\, e^{-i\frac{\pi}{3}} \left\{ e^{+i(z)} e^{+i\frac{5\pi}{12}} \right\}, \; \varepsilon \to 0+ \end{aligned} \qquad (5\ 3\text{-}11)$$

on the Stokes line S_2 (see Fig 5 1) After gathering terms,

$$\begin{aligned} y(t) &\sim T_1 \sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{-1/4}\, e^{-i\left\{ z + \frac{5\pi}{12} + \pi + \frac{\pi}{2} \right\}} \\ &\quad + \frac{T_1}{2}\sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{-1/4}\, e^{+i\left\{ z + \frac{5\pi}{12} - \frac{\pi}{3} - \frac{\pi}{2} \right\}}, \quad \varepsilon \to 0+ \end{aligned} \qquad (5\ 3\text{-}12)$$

$\Rightarrow$

$$y(t) \sim T_1 \sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{-1/4} \left\{ e^{-iz}\, e^{-i\frac{23\pi}{12}} + \frac{1}{2} e^{+iz}\, e^{-i\frac{5\pi}{12}} \right\}, \quad \varepsilon \to 0+ \qquad (5\ 3\text{-}13)$$

$\Rightarrow$

$$y(t) \sim T_1 \sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{-1/4} \left\{ -i\, e^{-iz}\, e^{-i\frac{5\pi}{12}} + \frac{1}{2} e^{+iz}\, e^{-i\frac{5\pi}{12}} \right\}, \quad \varepsilon \to 0+ \qquad (5\ 3\text{-}14)$$

$\Rightarrow$

$$y'(t) \sim T_1 \sqrt{\frac{3\varepsilon}{\pi}}\,(t+\lambda)^{+1/4}\, \frac{i}{\varepsilon} \left\{ +i\, e^{-iz}\, e^{-i\frac{5\pi}{12}} + \frac{1}{2} e^{+iz}\, e^{-i\frac{5\pi}{12}} \right\}, \quad \varepsilon \to 0+ \qquad (5\ 3\text{-}15)$$

Now, taking $t = 0$, we get

$$y(0) \sim T_1 \sqrt{\frac{3\varepsilon}{\pi}}\,(\lambda)^{-1/4} \left\{ -i\, e^{-iz}\, e^{-i\frac{5\pi}{12}} + \frac{1}{2} e^{+iz}\, e^{-i\frac{5\pi}{12}} \right\}, \quad \varepsilon \to 0+ \qquad (5\ 3\text{-}16)$$

where now $z = 2/(3\varepsilon)\,\lambda^{3/2}$ and

$$y'(0) \sim T_1 \sqrt{\frac{3\varepsilon}{\pi}}\,(\lambda)^{+1/4}\,\frac{\imath}{\varepsilon}\left\{+\imath\, e^{-\imath z}\, e^{-\imath\frac{5\pi}{12}} + \frac{1}{2}\, e^{+\imath z}\, e^{-\imath\frac{5\pi}{12}}\right\}, \qquad \varepsilon \to 0+ \qquad (5\,3\text{-}17)$$

Substituting (5 3-16) and (5 3-17) into our boundary condition at the origin (3 5-7), we get

$$\begin{aligned} &\imath\,\lambda^{1/4}\left[+\imath\, e^{-\imath z}\, e^{-\imath\frac{5\pi}{12}} + \frac{1}{2}\, e^{+\imath z}\, e^{\frac{-\imath 5\pi}{12}}\right] \\ &+P\,\lambda^{-1/4}\left[-\, e^{-\imath z}\, e^{-\imath\frac{5\pi}{12}} + \frac{1}{2}\, e^{+\imath z}\, e^{-\imath\frac{5\pi}{12}}\right] \to 0, \end{aligned} \qquad (5\,3\text{-}18)$$

as $\varepsilon \to 0+$, where

$$P = \frac{1}{2\left[\frac{1}{m+1} + \frac{\beta}{\delta+1}\right]} \qquad (5\,3\text{-}19)$$

So (5 3-18), becomes

$$\lambda \sim -P^2\left(\frac{1+\imath/2e^{+2\tau}}{1-\imath/2e^{+2\tau}}\right)^2, \qquad \varepsilon \to 0+, \qquad (5\,3\text{-}20)$$

where

$$\tau = \frac{\imath}{\varepsilon}\int_{-\lambda}^{0}[\lambda+s]^{1/2}ds \qquad (5\,3\text{-}21)$$

Here we use the fact that

$$\int_{-\lambda}^{0} J(s)ds - \tau \to 0, \qquad \varepsilon \to 0+ \qquad (5\,3\text{-}22)$$

Now, taking the leading terms of (5 3-20), we get

$$\lambda \sim -P^2\left(1+\imath 2e^{+2\tau}\right), \qquad \varepsilon \to 0+ \qquad (5\,3\text{-}23)$$

Thus to obtain our expression for the radiation loss, we need the following formula derived in [11] That is

$$\lambda = -P^2 - n!\,\varepsilon^n/(2P)^n + o(\varepsilon^n) \qquad (5\,3\text{-}24)$$

In our particular case $n = 1$, which implies $\lambda = -P^2 - \varepsilon/2P$ So from (5 3-21) and (5 3-24)

$$\tau = \frac{2\imath}{3\varepsilon}\lambda^{3/2} = \frac{2\imath}{3\varepsilon}\left[-P^2 - \varepsilon/2P\right]^{3/2}, \tag{5 3-25}$$

$\Rightarrow$

$$\tau = -\frac{2P^3}{3\varepsilon}\left[1 + \frac{3\varepsilon}{4P^3} + \quad \right] \tag{5 3-26}$$

Here we take $(-1)^{3/2}$ to be $e^{-\imath\frac{3\pi}{2}} = +\imath$ Then

$$e^{+2\tau} = e^{\frac{-4P^3}{3\varepsilon}\left[1+\frac{3\varepsilon}{4P^3}+ \quad \right]} \tag{5 3-27}$$

and we have to leading order

$$e^{+2\tau} = e^{\frac{-4P^3}{3\varepsilon}-1} \tag{5 3-28}$$

Therefore, from (5 3-23)

$$Im(\lambda) \sim -\frac{2P^2}{e}exp\left\{\frac{-4P^3}{3\varepsilon}\right\} \tag{5 3-29}$$

where $P = P(\beta, \sigma, m)$ is defined by (5 3-19) and contains the shape parameters

5.4 Power Index Profiles.

In section 2 4 we stated how various authors, [19],[3], [11], used asymptotic methods to solve Paris & Wood's model

$$y''(x) + (\lambda + \varepsilon\, x^n)y(x) = 0, \tag{5 4-1}$$

with boundary condition

$$y'(0) + hy(0) = 0, \tag{5 4-2}$$

for the respective cases $n = 1$, $n = 2$, $n > 2$ and how through doing so they obtained expressions for the radiation loss, $Im\lambda$ In this thesis, we showed for the particular case $n = 1$, how by simply rescaling our model (3 5-6)-(3 5-7), similar in nature to

Paris & Wood's, into

$$\varepsilon^2 y''(t) + (\lambda + t)y(t) = 0, \qquad (5\ 4\text{-}3)$$

with boundary condition

$$\varepsilon y'(0) + Py(0) = 0, \qquad (5\ 4\text{-}4)$$

we can obtain without the need for asymptotic matching, an expression for the radiation loss given by

$$Im\,\lambda \sim -\frac{2h^2}{e}\,exp\left\{\frac{-4h^3}{3\varepsilon}\right\} \qquad (5\ 4\text{-}5)$$

Similarily, for the cases $n = 2$, $n > 2$ we can apply Hu & Cheng's method to (5 4-1) and obtain a general expression for the radiation loss, $Im\lambda$ This can be achieved by substituting $x = \varepsilon^q t$ into (5 4-1) to get

$$\varepsilon^2 y''(t) + (\lambda^* + t^n)y(t) = 0, \qquad (5\ 4\text{-}6)$$

with boundary condition

$$\varepsilon y'(0) + Py(0) = 0, \qquad (5\ 4\text{-}7)$$

where $\lambda^* = \lambda\,\varepsilon^{\frac{-2(2n+1)}{n+2}}$

So, if we apply Hu & Cheng's method, outlined in section 5 3 to (5 4-6), this time taking into account any asymptotic matching that occurs and using the fact that $\gamma = 3/2$ and

$$\lambda = -P^2 - n!\,\varepsilon^n/(2P)^n + o(\varepsilon^n), \qquad (5\ 4\text{-}8)$$

we can obtain a general formula for the radiation loss for $n > 1$, given by [4]

$$Im\,\lambda \sim -2h^2\,exp\left\{\frac{-2\,h^{(n+2)/n}}{\varepsilon^{1/n}}\;\frac{\Gamma(1+1/n)\,\Gamma(3/2)}{\Gamma(\frac{3}{2}+\frac{1}{n})}\right\} \qquad (5\ 4\text{-}9)$$

[4] see (2 4-3) or [11]

For example if we take the case $n = 2$, from (5 4-9) we get

$$Im\,\lambda \sim -2h^2\, exp\left\{\frac{-\,2\,h^2}{\varepsilon^{1/2}}\,\frac{\Gamma^2(\frac{3}{2})}{\Gamma(2)}\right\} \tag{5 4-10}$$

$$\sim -2h^2\, exp\left\{\frac{-\,2\,h^2}{\varepsilon^{1/2}}\,\frac{\pi}{4}\right\} \tag{5 4-11}$$

$$\sim -2h^2\, exp\left\{\frac{-\,\pi\,h^2}{2\varepsilon^{1/2}}\right\} \tag{5 4-12}$$

which is what Brazel, Lawless & Wood obtained in [3] So not only does Hu & Cheng's method offer us a method to obtain the radiation loss to equations of type

$$\varepsilon^2 y''(t) \,+\, Q(t,\lambda)\, y(t) \,=\, 0, \tag{5 4-13}$$

with $Q(t,\lambda) \;=\; \lambda + t^n$, for any $n \in Z^+$, it also offers us a method to obtain the radiation loss for more arbitrary functions as long as they adhere to conditions (4 3-2)

Chapter 6

CONCLUSION.

We have addressed in this thesis the problem of modelling radiation loss from weakly-guiding optical fibres with realistic refractive index profiles We have seen that while the model of Kath and Kriegsmann is close to the physics, an explicit mathematical solution is not possible On the other hand, the idealised models described in Chapter 2, while solvable explicitly in terms of special functions, can rightly be criticised on physical grounds and cannot be used for many real fibres such as the W-shaped profile which is the subject of our work

In Chapter 3 we have used the methods of Burzlaff and Wood to develop a new model for fibres with W-shaped profiles While lacking some physical features, it is still tolerably realistic in containing the shape parameters of the refractive index profile in the boundary condition at the origin and hence in the eventual solution for the eigenvalue parameter, whose imaginary part corresponds to the rate of energy loss

Our work then led us to consider methods applicable to differential equation models with a more general refractive index profile in the potential In Chapter 4 we have described recent results of Hu and Cheng, Gingold and Kruskal We have explained in Chapter 5 how Hu and Cheng developed a formula for the imaginary part of the eigenvalue for general potentials This is a novel approach which could lead to further results outside this thesis For the potential considered in our model of the W-shaped profile fibre, a simplification of the method of Hu and Cheng allowed us to construct explicit Hankel function solutions of the differential equation Taking into account the Stokes phenomenon for the Hankel function, we arrived at the estimate of the imaginary part of the eigenvalue

We showed finally that direct substitution in the formula of Hu and Cheng produced the same result in this case, as it did for the power index profiles considered by Brazel, Lawless and Wood and by Liu and Wood

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